Lorentz invariance and causality

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$$c^2 t'^2 \underbrace{- x'^2 - y'^2 - z'^2}$$

$$= \gamma^2 (ct - \beta x)^2 - \gamma^2(-\beta tc + x)^2 - y^2 - z^2$$

$$= \gamma^2 (c^2 t^2 \cancel{- 2\beta x c t} + \beta^2 x^2) - \gamma^2 (x^2 \cancel{- 2\beta t c x} + \beta^2 t^2 c^2)$$
$$- y^2 - z^2$$

$$= \gamma^2 (c^2 t^2 - c^2 \beta^2 t^2) + \gamma^2 (\beta^2 x^2 - x^2) - y^2 - z^2$$

$$= c^2 \frac{t^2 - \beta^2 t^2}{1 - \beta^2} + x^2 \frac{\beta^2 - 1}{1 - \beta^2} - y^2 - z^2$$

$$= c^2 t^2 - x^2 - y^2 - z^2$$

$$= s^2$$

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

$E_1 = t_1, x_1, y_1, z_1$

$E_2 = t_2, x_2, y_2, z_2$

Suppose $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$,

$\Delta s^2 = c^2 \Delta t^2$

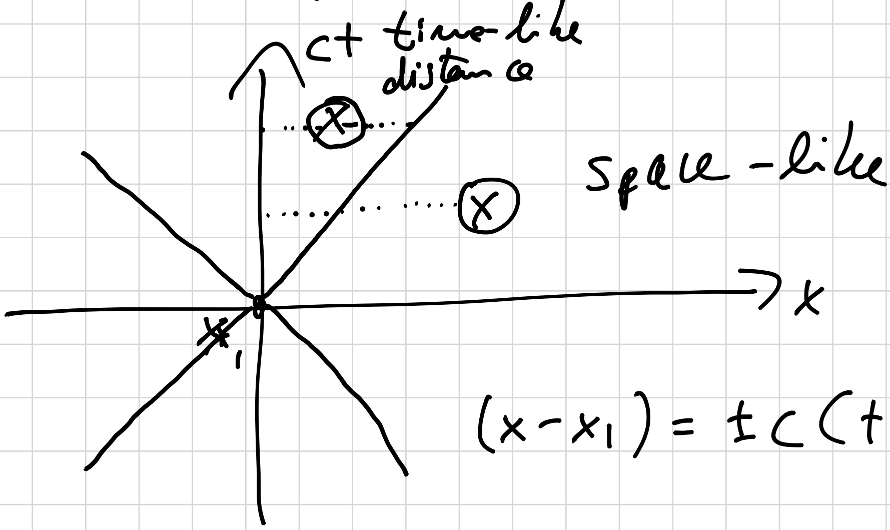
$\frac{\sqrt{\Delta s^2}}{c} = \Delta t_0$ proper time

if $\Delta s^2 < 0$, there is no frame in which the two events occur at the same place therefore, the two events are separated by a space-like distance

Suppose $t_1 = t_2$,

$\Delta s^2 = -\Delta x^2 - \Delta y^2 - \Delta z^2 = -l^2$

$l = \sqrt{-\Delta s^2}$ proper length



$(x - x_1) = \pm c(t - t_1)$

$t_1, x_1, y_1 = 0, z_1 = 0$

Energy-momentum four vector Lorentz transformations

$$\tau = \frac{1}{c}\sqrt{c^2t^2 - x^2 - y^2 - z^2} \quad \text{invariant}$$

$$d\tau = \frac{1}{c}\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2} \quad \text{invariant}$$

$$\frac{d\tau}{dt} = \frac{1}{c}\sqrt{c^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2}$$

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{u^2}{c^2}}$$

$$\frac{dx}{dt} = \vec{u}$$

$$\frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt}\,\frac{dt}{d\tau} = \vec{u}\,\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\vec{p} = \frac{m_0\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}} = m_0\frac{d\vec{x}}{d\tau}$$

$$= \cancel{m_0\vec{u}}$$

$$E = m_0\gamma c^2 = m_0\frac{c^2}{\sqrt{1 - u/c}} = m_0 c\,\frac{d(ct)}{d\tau}$$

$$\frac{E}{c} = m_0\frac{d(ct)}{d\tau}$$

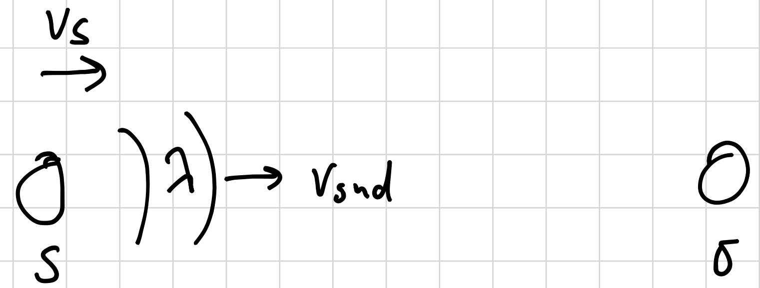
Because $(ct, \vec{x})$ constitutes a Lorentz 4-vector, $(\frac{E}{c}, \vec{p})$ is also a Lorentz 4-vector, which means that it transforms from one reference frame to another by the Lorentz transformation

$$s^2 = c^2t^2 - \vec{x}^2$$

$$m_0^2 = \frac{E^2}{c^2} - \vec{p}^2$$

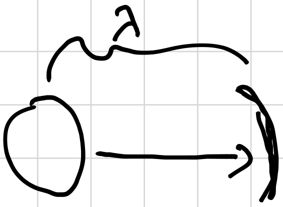
$$E^2 = m_0^2 c^4 + p^2 c^2$$

Doppler



$$\lambda \cdot f = v_{snd}$$

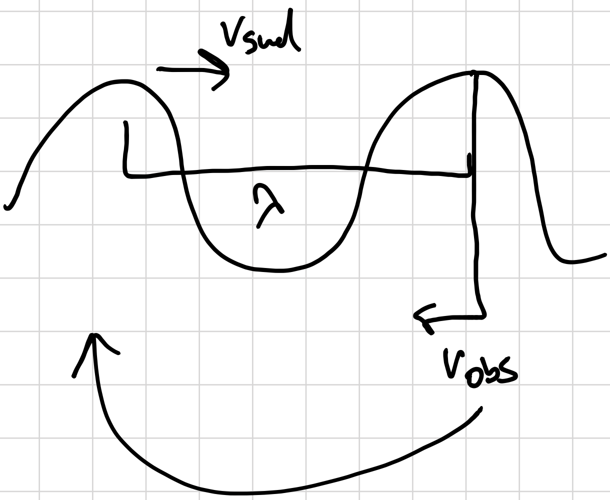
$$\lambda = \frac{v_{snd}}{f}$$



$$\lambda' = \lambda - v_s t = \lambda - \frac{v_s}{f} = \lambda\left(1 - \frac{v_s}{\lambda f}\right)$$

$$= \lambda\left(1 - \frac{v_s}{v_{snd}}\right)$$

$$f' = f\,\frac{1}{1 - \frac{v_s}{v_{snd}}} = f\,\frac{v_{snd}}{v_{snd} - v_s}$$



$$\lambda = t\,(v_{obs} + v_{snd})$$

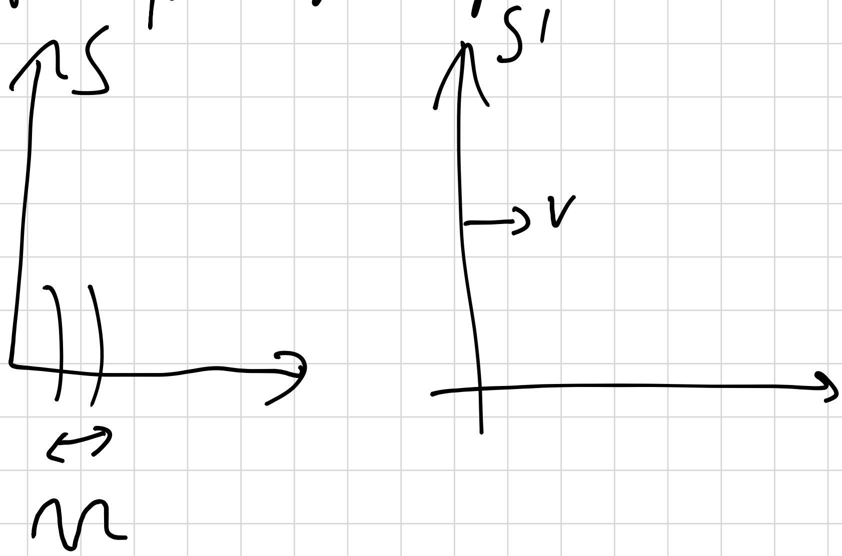
$$\frac{\lambda}{v_{obs} + v_{snd}} = t = \frac{1}{f'}$$

$$f' = \frac{v_{obs} + v_{snd}}{\lambda} = \frac{f}{v_{snd}}\left(v_{obs} + v_{snd}\right)$$

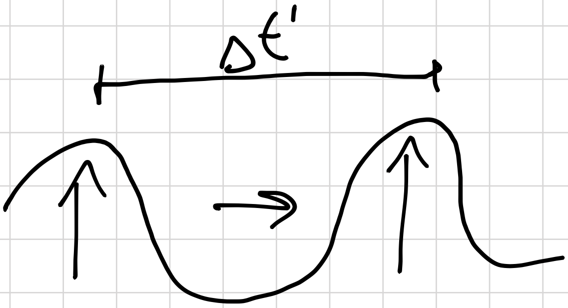
$$f' = f\,\frac{v_{obs} + v_{snd}}{v_{snd}}$$

$$f' = f \, \frac{V_{snd} + V_{obs}}{V_{snd} - V_s}$$

Doppler effect for light



$$\lambda' = c\Delta t' + v\Delta t' = (c+v)\Delta t'$$



$$\Delta t = \frac{1}{f}$$

$$\Delta t' = \gamma \Delta t = \gamma \frac{1}{f}$$

$$\lambda' = (c+v)\gamma \frac{1}{f} = (c+v)\gamma \frac{\lambda}{c}$$

$$= \frac{c+v}{\sqrt{1-\frac{v^2}{c^2}}} \frac{1}{c} \lambda = \frac{1+\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} \lambda$$

$$\lambda' = \frac{1+\frac{v}{c}}{\sqrt{\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)}}\,\lambda$$

$$1-x^2 = (1+x)(1-x)$$

$$\boxed{\lambda' = \frac{\sqrt{1+\frac{v}{c}}}{\sqrt{1-\frac{v}{c}}}\,\lambda}$$

$$\boxed{f' = \frac{\sqrt{1-\frac{v}{c}}}{\sqrt{1+\frac{v}{c}}}\,f}$$

for two frames moving <u>away</u> from each other